REDUCTION OF NONHOLONOMIC VARIATION PROBLEMS TO ISOPERIMETRIC ONES AND CONNECTIONS IN PRINCIPAL BUNDLES

A. M. Vershik and O. A. Granichina

1. Introduction

By a nonholonomic variation problem on a manifold with given distribution, i.e., on a nonholonomic manifold, is meant a variation problem on the class of curves tangent to the distribution. If the distribution is integrable, then the problem reduces to an ordinary variation problem on some boundary or initial conditions. Now if the distribution is non-integrable, for example, absolutely nonholonomic, then we have a general Lagrange problem in which the Langrange multipiers generally depend on time. Attempts to reduce this problem to an unconditional variation problem have been undertaken a number of times (mainly in applied papers), but up to now there have apparently not been any precise formulations. Later we shall show how for some nonholonomic distributions, namely horizontal distributions of a connection in principal or associated bundles, one can reduce the problem to an isoperimetric problem, thus reducing the number of conditions to a finite number. The relation between the two problems can be used in both directions; in particular, classical isoperimetric problems acquire a new interpretation. We restrict ourselves here to the case of principle bundles but the scheme can be used in general. Namely, the only thing actually required of the nonholonomic distribution is invariance with respect to a group the orbits of whose action are transverse to the distribution. This case occurs very often in applications, in particular in problems on nonholonomic Lie groups which are considered later. By isoperimetric problem we have in mind a variation problem on a set of curves of given smoothness (with some initial-boundary conditions) and forming a manifold of finite codimension in the space of all curves. The number of Lagrange multipliers in these problems is finite.

In Sec. 2 we formulate a nonholonomic variation problem and a certain isoperimetric problem and show their equivalence. Simultaneously we forumulate the corresponding Cauchy problem which is needed for the construction of nonholonomic geodesic flows. In Sec. 3 we study the group case. The corresponding problems on the Heisenberg group and Engel group (examples of nilpotent groups) are considered in Sec. 4. The case of the Engel group for which the Euler–Lagrange equations can be integrated in terms of elliptic functions is especially interesting.

2. Equivalence Theorem

Let $P \overset{\pi}{\to} B$ be a principal bundle whose fiber is a Lie group H with Lie algebra H, $\mathfrak{H}$. We define a connection in P. Let ω be the $\mathfrak{H}$-valued connection base-form, $\bar{\omega}$ the connection form in TP corresponding to it, ∇ be the covariant differentiation operator. By Π_γ we denote the operation of parallel transport of the fiber of the principal bundle P over the initial point of the curve γ from B to the fiber over the end-point of the curve γ; Π_γ is an element of the group H (more precisely an element of the holonomy group). Subsequently, we consider the case when the connection is complete, i.e., the holonomy group coincides with H. This can be expressed in another way as follows: The horizontal distribution of the connection $Z \subset TP$ is absolutely nonholonomic; according to the Rashevskii–Chow theorem [1] this means that any two points of P (it suffices to make the requirement for points of one fiber only) can be joined by a horizontal curve. We shall assume that on the horizontal subspace there is given a Finsler (Riemannian) metric or in the more general case on Z there is defined a Lagrangian $\mathcal{L}$ which gives the action functional

$$\mathcal{F}(\bar{\gamma}) = \int_{\bar{\gamma}} \mathcal{L}.$$

Leningrad State University. Translated from Matematicheskie Zametki, Vol. 49, No. 5, pp. 37-44, May, 1991. Original article submitted February 15, 1990.

Problem I (nonholonomic variation problem). Find an extremal of the functional $\mathscr{G}$ on the set of horizontal curves with given ends.

This is a Lagrange problem with an infinite number of conditions since it is required that at each point the velocity vector of the curve sought belong to the horizontal subspace at the given point.

We formulate the Cauchy problem corresponding to Problem I.

For this we introduce the mixed bundle Ken Z = $Z \oplus Z^\perp$ (cf. [3]), where $Z^\perp \subset T^*P$. We shall interpret (cf. [2, p. 330[; [3]) the subbundle $Z^\perp$ as the space of Lagrange multipliers. We define initial conditions as follows: $p \in P$ is the initial point, $z \in Z$ is an admissible vector from T_pP, $\lambda \in Z^\perp$ is the initial value of the Lagrange multiplier.

Problem Ia. Construct a field of horizontal extremals of the functional $\mathscr{G}$ in the mixed bundle Ken Z.

If one passes to the cotangent bundle T^*P, then the corresponding flow in Problem Ia is Hamiltonian.

Let us now assume that the Lagrangian $\mathscr{L}$ is invariant with respect to the fiberwise action of the group H or P so that it can be considered as a functional on the base. We denote it by the same letter $\mathscr{L}$ and the corresponding action functional by $\mathscr{G}$.

We consider the isoperimetric problem.

Problem II. Find a piecewise-smooth curve $\gamma: [0, 1] \to B$ on the base B which is minimizing in the sense of the metric (generally extremal) parallel transport along which carries the fiber over the initial point of the curve γ in a given way into the fiber over the end-point $\gamma(1)$, i.e.,

$$\Pi_\gamma \gamma(1) = g, \quad g \in \pi^{-1}(\gamma(1)). \tag{1}$$

One can write this condition more explicitly in the group case (cf. below). One can say that one seeks a shortest curve in a "given holonomy class." Problem II is isoperimetric in the sense indicated above since the number of conditions on the curve defined by the holonomy class is finite.

We recall (cf. [4, p. 69]) that a lift of the curve γ from the base B is the unique horizontal curve $\tilde{\gamma} = L\gamma$ in P starting at a given point and such that $\pi\tilde{\gamma} = \gamma$.

The following theorem holds.

THEOREM. The set of lifts of an extremal in Problem II is the same as the set of extremals of Problem I.

Proof. We write the Euler–Lagrange equations for both problems and compare them. First, we consider Problem II and we form the Lagrange function for it:

$$\mathscr{H}(\lambda) = \mathscr{L}(\dot\gamma) + \langle\lambda, \omega(\dot\gamma)\rangle \equiv \mathscr{L}(\dot\gamma) + a_\lambda(\dot\gamma),$$

where $\lambda: Z \to \mathfrak{H}^*$ ($\mathfrak{H}^*$ -valued form on Z), a_λ is a 1-form depending linearly on λ. The Euler–Lagrange equation has the form

$$\mathscr{E}(\dot\gamma) = [(da_\lambda)_x]^T \llcorner \dot\gamma - \langle\dot\gamma, d_x a_\lambda\rangle, \tag{2}$$

where $\mathscr{E}$ is the Euler–Lagrange differential operator of the unconditional problem. For fixed initial conditions λ can be found from the conditions on the right end of the curve.

We write the Euler–Lagrange equation for Problem I:

$$\mathscr{L}(\dot{\tilde\gamma}) = [(d\tilde{a}_\lambda)_x]^T \llcorner \dot{\tilde\gamma} - \langle\dot{\tilde\gamma}, d_x\tilde{a}_\lambda\rangle, \tag{3}$$

where $\tilde{a}_\lambda(\dot{\tilde\gamma}) = \langle\lambda, \tilde\omega(\dot{\tilde\gamma})\rangle$.

We consider the right side of (3). From the properties of a principal bundle (cf. [4, p. 70]) one can choose an open covering $\{U_\alpha\}$ of the base B: $\forall \pi^{-1}(U_\alpha)\ \exists$ an isomorphism j_α: $x \to (\pi(x), \varphi_\alpha(x))$ onto $U_\alpha \times H$. We define a correspondence between curves in the base B and curves in the bundle P as follows. Let γ be a curve in B and associate with it the piecewise-smooth curve $\check\gamma$ in P: $\check\gamma(t) = j_\alpha^{-1}(\gamma(t), e)$, $\forall t: \gamma(t) \in U_\alpha$. By construction of the curve $\check\gamma$ we have $\pi(\check\gamma) = \gamma$. Since $\tilde\gamma = L\gamma$ is a horizontal curve, in the space TP

$$\tilde{\dot{\gamma}} = \dot{\tilde{\gamma}} - \nu\dot{\tilde{\gamma}}, \tag{4}$$

where $\nu\dot{\gamma}$ is the vertical component of the vector field $\dot{\gamma}$ (cf. [4, p. 68]; [5, p. 321]). We note that by the choice of ω, $\tilde{\omega}$ and the Lagrangian $\mathscr{L}$

$$\begin{gathered}\tilde{a}_\lambda(\tilde{\dot{\gamma}}) = a_\lambda(\dot{\gamma}),\\ [(d\,\tilde{a}_\lambda)_x]^{\gamma} \lfloor\, \nu\dot{\tilde{\gamma}} - \langle \nu\dot{\tilde{\gamma}}, d_x\tilde{a}_\lambda\rangle = 0.\end{gathered} \tag{5}$$

Substituting (4) into (3), considering (5), we get that the right side of (3) is the same as the right side of (2). Thus, we have found that (2) and (3) are the same.

We turn our attention to the fact that the Lagrange multipliers in Problem I are actually constant (more precisely H-invariant) which lets us reduce Problem I to Problem II. The theorem is proved.

It is helpful to formulate the Cauchy problem corresponding to Problem II.

Problem IIa. Construct a field of extremals of the functional $\mathscr{F}$ in the class of curves on B with initial conditions: $u \in B$ is the initial point, $f \in T_uB$ is the initial velocity vector, and the additional initial condition

$$\nabla_{\dot{\gamma}}\dot{\gamma}\,|_{t=0} = h_{(u)},$$

where h is a section of the bundle $P \xrightarrow{H} B$. The field of extremals of the functional $\mathscr{F}$ depends on the choice of the section h. By changing h one can get all extremals of Problem II. The connection between Problems Ia and IIa can be described by a theorem analogous to the one given above.

3. Group Case

Let G be a Lie group of dimension n, H be a closed subgroup of it ($\dim H = n - p$), $\mathfrak{G}$ be the Lie algebra of the group G. We consider $\mathfrak{H} = T_eH$.

We define the principal bundle G over the group H:

$$G \xrightarrow{H} G/H = B, \tag{6}$$

where π is the canonical projection. For simplicity we assume that the bundle is trivializable. We fix a trivialization and denote by i the map $i\colon B \to G$, $i(z) = (z, e)$. We introduce a connection in the bundle as follows. Let $\omega\colon TB \to \mathfrak{H}$ be the connection base form on B. By $\tilde{\omega}_g$ we denote its lift at the point $g \in G$. Then $\operatorname{Ker}\tilde{\omega}_g = Z_g$ is the horizontal subspace at the point g. We note that generally the connection form is not left-invariant, for example, this is not so for the Engel group.

Suppose given on B a Riemannian (or generally Finsler) metric $\|\cdot\|$ [2, p. 329]; [3].

Problem I' (on nonholonomic geodesic). Let $g_0, g_1 \in G$. Find a shortest horizontal curve with respect to the metric $\tilde{\rho}$ in G joining the points g_0 and g_1.

Without loss of generality we assume that $g_0 = e$, $g_1 = g$.

Making the arguments of Sec. 2 in detail we write (1) explicitly in terms of forms. We shall assume that H is imbedded in G by virtue of a choice of trivialization of the bundle. Let w(t) be a curve in the structural group H: $w(0) = e$; then (1) assumes the form (cf. [4, p. 73])

$$w^{-1}(t)\,\dot{w}(t) = -\omega(\dot{\gamma}(t)), \quad \Pi_\gamma = w(1), \tag{7}$$

(Maurer–Cartan equation or logarithmic derivative equation). Integration of it with the initial condition $u \in B$ gives the image of u under parallel transport. By the thoerem of Sec. 2 Problem I is equivalent to the following isoperimetric problem.

Problem II'. Find a shortest curve γ in B in the sense of the metric $\|\cdot\|$ joining the points e and $u = \pi(g)$ under the isoperimetric condition (7).

We pass from Problem II' to the corresponding Cauchy problem in B.

Problem IIa'. On the manifold B with Riemannian metric $\|\cdot\|$ find

$$\inf \int_0^t \|\dot{\gamma}(t)\| dt,$$

among all curves $\gamma: [0, 1] \to B$, $\gamma(0) = e$, $\dot{\gamma}(0) = f$, $\nabla_{\dot{\gamma}}\dot{\gamma}|_{t=0} = h \in \mathfrak{H}$.

If one assumes that the initial acceleration vector h = 0 (in the case of a Riemannian metric), then the solution of Problem IIa' is a geodesic; for general h we shall call the corresponding class of curves starting from zero pseudogeodesic.

To solve Problem II' we form the Lagrange function:

$$\mathcal{M}(\lambda) = \|\dot{\gamma}\| + \langle \lambda, \omega(\dot{\gamma})\rangle.$$

Then the Euler–Lagrange equation has the form: $\nabla_{\dot{\gamma}}\dot{\gamma} = \ell_\lambda(\dot{\gamma}) = \ell_\lambda(\dot{\gamma})$, where ℓ_λ is a family of scalar forms on B depending linearly on the Lagrange multipliers λ (equation of motion in a magnetic field). Here λ is found from the initial conditions and (7).

4. Examples of Integrable Problems

4.1. Heisenberg Group. Let N be the three-dimensional Heisenberg group. We consider its Lie algebra which one can realize by vector fields on ^{3}

$$e_1 = \frac{\partial}{\partial x_1} + \frac{x_2}{2}\frac{\partial}{\partial x_3},$$
$$e_2 = \frac{\partial}{\partial x_2} - \frac{x_1}{2}\frac{\partial}{\partial x_3},$$
$$e_3 = \frac{\partial}{\partial x_3}.$$

We define a principal bundle $N \cong \mathbb{R}^3$ over the group $[N, N] \cong \mathbb{R}$:

$$N \xrightarrow[\pi]{[N,N]} N|_{[N,N]} = B.$$

We introduce a connection on this bundle. As the horizontal space at the point $g \in N$ we take $Z_g = \mathrm{Lin}\{e_1, e_2\}$. This connection is left-invariant since $[N, N]Z \subset Z[N, N]$. Let a Riemannian metric be introduced in the space B. We formulate Problems I and II for the given case.

I (on the group N). Find a curve $\tilde{\gamma} = (x_1, x_2, x_3)$ realizing

$$\inf \int_0^1 (\dot{x}_1^2 + \dot{x}_2^2)\, dt,$$

under the conditions

$$x_i(0) = 0, \quad x_i(1) = \bar{x}_i, \quad i = 1, 2, 3, \quad \dot{x}_3 = \frac{1}{2}(\dot{x}_1 x_2 - \dot{x}_2 x_1).$$

II (on the set B). Find a curve $\gamma = (x_1, x_2)$ realizing

$$\inf \int_0^1 (\dot{x}_1^2 + \dot{x}_2^2)\, dt,$$

under the conditions

$$\begin{gathered}\int_0^1 (\dot{x}_1 x_2 - \dot{x}_2 x_1)\, dt = c_3, \\ x_i(0) = 0, \quad x_i(1) = \bar{x}_i, \quad i = 1, 2.\end{gathered} \tag{8}$$

Problems I and II are equivalent; one can find their solution in [2]. We note that the form in (8) is the area form so (8) is the condition that the area bounded by the curve γ and the segment joining the initial and terminal point is preserved. Hence Problem I reduces to the classical isoperimetric problem on B. For a metric of the form $|\dot{\gamma}| = |\dot{x}_1| + |\dot{x}_2|$ formulas for extremals are obtained in [6], [7]. In [8] the corresponding problems for the multidimensional Heisenberg group are considered. For an arbitrary metabelian Lie group Problems I and II are given in [9].

4.2. Engle Group. Let $\mathfrak{A}$ be a four-dimensional nilpotent Lie algebra with basis ξ_1, ξ_2, ξ_3, ξ_4, two generators ξ_1, ξ_2, and generating relations $[\xi_1, \xi_2] = \xi_3$, $[\xi_2, \xi_3] = \xi_4$, the remaining brackets being equal to zero. For this group solutions of Problems I and II can be written with the help of elliptic functions.

With the basis one can associate vector fields

$$\xi_1 = \frac{\partial}{\partial x_1}, \quad \xi_2 = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3} + x_1 x_2 \frac{\partial}{\partial x_4},$$
$$\xi_3 = \frac{\partial}{\partial x_3} + x_2 \frac{\partial}{\partial x_4}, \quad \xi_4 = \frac{\partial}{\partial x_4}.$$

The Lie group G of the algebra $\mathfrak{A}$ is the set of upper-triangular matrices of the form

$$\begin{pmatrix} 1 & a & c & d \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (a, b, c, d) \in \mathbb{R}^4.$$

We consider the principal bundle

$$G \xrightarrow[\pi]{H} G/H = B,$$

where H is the group generated by matrices of the form

$$\begin{pmatrix} 1 & 0 & c & d \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (c, d) \in \mathbb{R}^2.$$

We introduce a connection on the principal bundle by taking as horizontal subspace at the point $g \in G$, $Z_g = \mathrm{Lin}\{\xi_1, \xi_2\}$. We assume that on B a Euclidean metric is introduced. We consider two equivalent problems.

I. In the space G find a curve $\bar{\gamma}$ realizing

$$\inf\left\{\int_0^1 \|\dot{\bar{\gamma}}\| dt \colon \bar{\gamma}(0) = 0,\ \bar{\gamma}(1) = (a_i)_{i=1}^4,\ \dot{\bar{\gamma}} \in Z\right\}.$$

II. In the space B find a piecewise-smooth curve $\gamma = (x_1, x_2)$ realizing

$$\inf\left\{\int_0^1 \|\dot{\gamma}\| dt \colon \gamma(0) = 0,\ \gamma(1) = (x_1, x_2),\right.$$
$$\left.\int_\gamma x_1\, dx_2 = x_3,\quad \int_\gamma x_1 x_2\, dx_2 = x_4\right\}.$$

Defining constants λ_1, λ_2, a, $b\,(a > b)$ in the corresponding way for the boundary conditions, as a result of the solution we get:

$$y_1 = (|b| + |a|/2 + a/2)\, E(\mathrm{am}(u), k) - |a|\, u - (b + |a|)\, t/2,$$

$$y_2 = \begin{cases} \sqrt{|b|}\, \mathrm{cn}(u) & \text{for } a \geqslant 0, \\ \pm\sqrt{(|b| - |a|)\,\mathrm{cn}^2(u) + |a|} & \text{for } a < 0, \end{cases}$$

where

$$k = \frac{\sqrt{|b| - a/2 - |a|/2}}{\sqrt{|b| - a/2 + |a|/2}},$$
$$u = \pm K(k) - \frac{\sqrt{|b| + a/2 + |a|/2}}{2\sqrt{2}}\,(t - \sqrt{a/2 + |a|/2}).$$

Here we have used the notation for the elliptic integral of the second kind $E(\varphi, k)$ of the complete elliptic integral of the first kind $K(k)$, of the elliptic cosine $\mathrm{cn}(u)$ and amplitude $\mathrm{am}(u)$. The relation between the parameters a and b is established from the initial conditions $x_1(0) = x_2(0) = 0$. A solution of Problem II, x_1 and x_2 is given by

$$x_1 = y_1/2\lambda_2, \quad x_2 = (y_2 - y_2(0))\lambda_2.$$

In order to write a solution of Problem 1 it is necesssary to use the following equalities:

$$x_3(t) = \int_0^t x_1 \dot{x}_2\,dt = \frac{g y_2}{2\lambda_2^2} - \int_0^t \frac{y_2^3 + (a - b) y_2/2}{2\lambda_2^2}\,dt,$$

$$x_4(t) = \int_0^t x_1 x_2 \dot{x}_2\,dt = -\lambda_2 x_3/\lambda_2 - \frac{g y_2^2}{4\lambda_2^2} - \int_0^t \frac{y_2^4 + (a - b) y_2^2/2}{4\lambda_2^2}\,dt.$$

The problem for general Lie groups of degree of nilpotency 3 is given in [9]. Here (7) can be rewritten in the form of conditions that certain areas and volumes are preserved.

LITERATURE CITED

1. P. K. Rashevskii, "Joining any two points of a completely nonholonomic space by an admissible line," Uch. Zap. Mosk. Ped. Inst. im. K. Libknekhta. Ser. Fiz.-Mat., 3, No. 2, 83-94 (1938).
2. A. M. Vershik and V. Ya. Gershkovich, "Nonholonomic problems and the geometry of distributions," Appendix to the Russian translation of P. Griffiths, Exterior Differential Systems and Calculus of Variations, Mir, Moscow (1986).
3. A. M. Vershik and V. Ya. Gershkovich, "Nonholonomic dynamical systems. Geometry of distributions and variational problems," Itogi Nauki Tekh. Ser. Sovremennye Probl. Mat., Fund. Nap., 16, 1-91 (1987).
4. K. Nomidzu, Foundations of Differential Geometry [Russian translation], Vol. 1, Nauka, Moscow (1981).
5. S. Sternberg, Lectures on Differential Geometry [Russian translation], Mir, Moscow (1970).
6. O. A. Granichina, "Sphere on the Heisenberg group for a nonholonomic polysystem," in: What's New in Global Analysis: Topological and Geometric Methods of Analysis [in Russian], Voronezh (1989), pp. 141-146.
7. L. S. Kirillova, "Non-Riemannian metrics and the maximum principle," Dokl. Akad. Nauk UzbSSR, No. 7, 9-11 (1986).
8. O. A. Granichina, "Shortest paths on the multidimensional Heisenberg group," Dep. in VINITI, No. 6862-B89.
9. A. M. Vershik and O. A. Granichina, "Nonholonomic variational problems in bundles and isoperimetric ones," Dep. in VINITI, No. 1390.